Temperature Gradients in Turbulent Gas Streams: Effect of Viscous Dissipation on Evaluation of Total Conductivity

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Values of the total conductivity in a turbulently flowing air stream were originally reported without regard to the effects of viscous dissipation upon the results. By utilizing measurements of the shear taken at the time of the original study, the reported values of total conductivity were corrected for the effect of viscous dissipation. The effect of such dissipation upon the values of total conductivity is the greatest under conditions involving small temperature gradients and high Reynolds numbers.

A knowledge of the interrelation of momentum and energy transport in turbulent flow is a useful tool in many engineering applications. The early concepts of Reynolds (*21*), Prandtl (*20*), Taylor (*24*), and von Kármán (*15*) were substantial contributions to an understanding of this problem. Boelter and co-workers (*3*) reconsidered the Reynolds analogy and extended it somewhat. Many experimental studies and theoretical treatments of velocity distribution and viscous dissipation in turbulent flow have been made during the past decade (*1, 7, 8, 9, 10, 14, 22*).

In the period from 1947 to 1956 a series of measurements was made in the authors' laboratory (*5, 6, 13, 18, 19, 23*) for the purpose of evaluating the total conductivity in turbulent flow. In this paper there is presented a supplemental evaluation of experimental data obtained earlier (*13, 19*) from investigation of the eddy conductivity for air in a steady, nearly uniform stream flowing between parallel plates. This supplemental evaluation involves determination of the effect of viscous dissipation in the calculation of total conductivity.

ANALYSIS

Following von Kármán (*15*) one may define total conductivity as

$$\epsilon_c = \frac{\mathring{q}}{C_p \sigma}\left(\frac{dy}{dt}\right) \tag{1}$$

The total conductivity is based upon the local value of the thermal flux, which may or may not be directly related to the boundary fluxes.

Figure 1 illustrates schematically the flow of air between parallel plates. For the experimental work under consideration the upper plate was maintained at a higher temperature than the lower one. When the temperature of the two plates is different but uniform, the thermal transport from the upper to the lower plate is independent of downstream position, and for steady conditions it is possible to evaluate the total conductivity from Equation (1), provided the local thermal flux can be evaluated. The experimental data (*13, 19*) only provide information concerning the flux at the upper boundary. In the original evaluation of the experimental data (*13, 19*) the local values of the molecular thermal conductivity and viscosity were taken to be a function of the local temperature, and the local heat flux was assumed equal to the flux at the upper wall.

For steady, uniform, two-dimensional, incompressible flow the energy

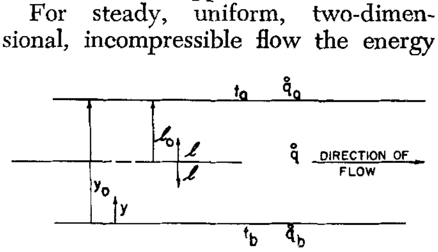

Fig. 1. Schematic arrangement of flow between parallel plates.

transport Equation (2) may be written in the form

$$\frac{d\mathring{q}}{dy} = \eta\Phi \tag{2}$$

Since the temperature decreases with a decrease in y, Equation (3) can be integrated to

$$\mathring{q} = \mathring{q}_a + \int_{y_0}^{y_0-y} \eta\Phi\, dy \tag{3}$$

For present conditions the dissipation function (2) is given by

$$\Phi = \left(\frac{\partial u_x}{\partial y}\right)^2 + \sum_{i=1}^{3}\sum_{j=1}^{3} \overline{\frac{\partial u'_i}{\partial x_j}\left(\frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\right)} \tag{4}$$

The definition of molecular shear for two-dimensional laminar flow is as follows:

$$\tau_m = \eta \frac{\partial u_x}{\partial y} \tag{5}$$

There results

$$\eta\Phi = \tau_m \frac{\partial u_x}{\partial y} + \eta \sum_{i=1}^{3}\sum_{j=1}^{3} \overline{\frac{\partial u'_i}{\partial x_j}\left(\frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\right)}$$
$$= \tau_m \frac{\partial u_x}{\partial y} + \eta\Phi_t \tag{6}$$

where

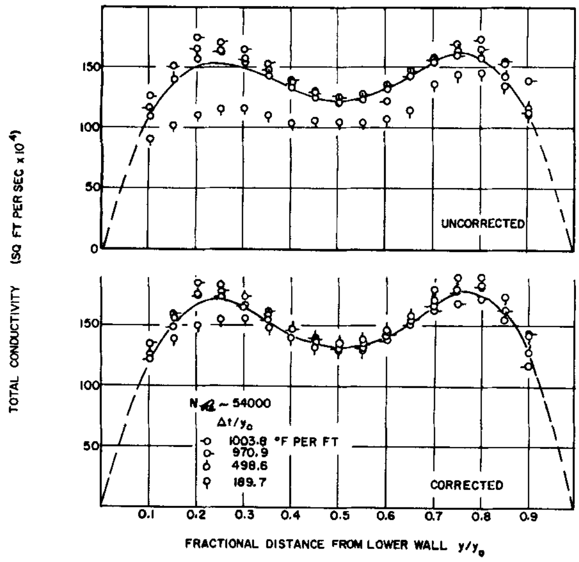

Fig. 2. Typical experimental values of total conductivity.

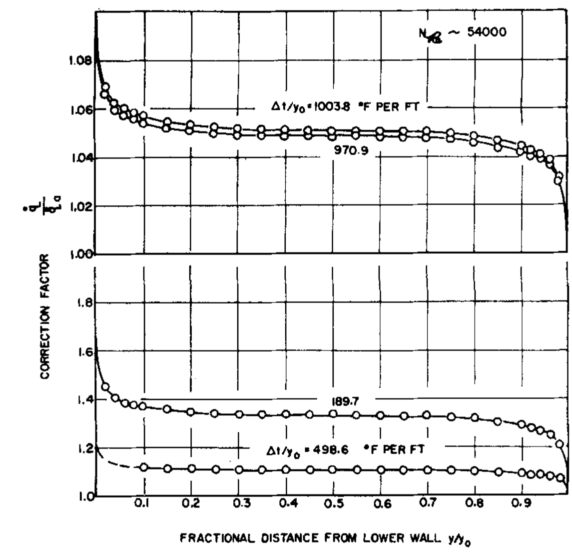

Fig. 3. Effect of temperature gradient and position upon viscous dissipation.

$$\Phi_t = \sum_{i=1}^{3}\sum_{j=1}^{3} \overline{\frac{\partial u'_i}{\partial x_j}\left(\frac{\partial u'_j}{\partial x_i}+\frac{\partial u'_i}{\partial x_j}\right)} \tag{7}$$

The information required to evaluate the second term in Equation (6) was not available for the experimental measurements under consideration. For the conditions of flow of interest it can be shown (*11*) that

$$-\rho\,\overline{u'_x u'_y}\,\frac{\partial u_x}{\partial y} = \eta \sum_{i=1}^{3}\sum_{j=1}^{3} \overline{\frac{\partial u'_i}{\partial x_j}\left(\frac{\partial u'_j}{\partial x_i}+\frac{\partial u'_i}{\partial x_j}\right)} - \left[\eta \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial}{\partial x_i}\,\overline{u'_j\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)} - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\,\overline{u'_i\,(P'+E)}\right] \tag{8}$$

In Equation (8) the term in brackets represents the difference between production and convection of turbulent energy. Data obtained by Laufer (*16, 17*) and recalculated by others (*11, 12, 25*) for channel and pipe flow indicate that at all positions in the flowing stream this term is small compared with the other two terms in Equation (8). Furthermore it can be shown (*12*) that the integral of this term over the entire flowing stream vanishes. Thus even though there exist local differences in the production and convection of the energy of turbulence, of the order of 5% of the average rate of viscous dissipation, no cumulative error in the integral of the dissipation function arises if these terms are neglected.

On the basis of the foregoing, if the turbulent shear is defined as the difference between the actual shear and the molecular shear defined by Equation (5), it follows that

$$\tau_t = \tau - \tau_m = -\rho\,\overline{u'_x u'_y} \tag{9}$$

The turbulent dissipation function may be written in the following way:

$$\Phi_t = \frac{\rho}{\eta}\,\overline{u'_x u'_y}\,\frac{\partial u_x}{\partial y} = \frac{1}{\eta}\,\tau_t\,\frac{\partial u_x}{\partial y} \tag{10}$$

Combining terms in Equations (6) and (10) one obtains

$$\eta\Phi = \tau_m\frac{\partial u_x}{\partial y} + \tau_t\frac{\partial u_x}{\partial y} = \tau\frac{\partial u_x}{\partial y} \tag{11}$$

TABLE 1. TOTAL CONDUCTIVITY* CORRECTED FOR VISCOUS DISSIPATION

Fractional distance from lower wall	Reynolds number 10,000	15,000	20,000	30,000	40,000	50,000	60,000
0.10	18.35×10^{-4}	30.97×10^{-4}	43.44×10^{-4}	66.98×10^{-4}	90.05×10^{-4}	112.48×10^{-4}	134.92×10^{-4}
0.15	26.32	41.43	56.37	84.58	112.21	139.09	165.97
0.20	32.25	48.63	64.83	95.40	125.34	154.47	183.60
0.25	35.04	51.48	67.74	98.43	128.49	157.74	186.99
0.30	34.23	49.98	65.55	94.94	123.74	151.75	179.75
0.35	32.99	47.56	61.96	89.14	115.67	141.68	167.58
0.40	31.59	44.98	58.21	83.18	107.65	131.45	155.25
0.45	30.40	43.17	55.80	79.63	102.98	125.69	148.40
0.50	30.44	42.94	55.31	78.65	101.52	123.77	146.01
0.55	31.24	43.79	56.19	79.61	102.54	124.86	147.17
0.60	32.76	45.93	58.93	83.49	107.54	130.94	154.34
0.65	34.37	48.58	62.63	89.14	115.12	140.39	165.66
0.70	35.22	51.17	66.96	96.75	125.93	154.32	182.71
0.75	36.03	53.39	70.55	102.95	134.69	165.56	196.43
0.80	32.54	50.05	67.35	100.01	132.01	163.14	194.27
0.85	27.45	43.82	60.01	90.57	120.50	149.62	178.74
0.90	18.00	31.71	45.26	70.85	95.91	120.30	144.68

* Total conductivity expressed in sq. ft./sec.
Average value of thermometric conductivity $K = 2.60 \times 10^{-4}$ sq. ft./sec.

TABLE 2. COMPARISON OF TOTAL CONDUCTIVITY CORRECTED AND UNCORRECTED FOR VISCOUS DISSIPATION

Fractional distance from center line	Total conductiivty,* sq.ft./sec.					
	Re = 20,000		Re = 40,000		Re = 60,000	
	Corrected	Uncorrected	Uncorrected	Corrected	Corrected	Uncorrected
0	55.3×10^{-4}	54.2×10^{-4}	101.5×10^{-4}	95.8×10^{-4}	146.0×10^{-4}	126.2×10^{-4}
0.2	58.6	58.4	107.6	103.0	154.8	139.4
0.4	66.3	68.0	124.8	118.2	181.2	164.6
0.6	66.1	68.0	128.7	120.5	188.9	171.6
0.8	44.4	42.1	93.0	82.2	139.8	119.6

* Thermometric conductivity, $K = 2.60 \times 10^{-4}$ sq. ft./sec.

Combining Equations (11) and (3) one obtains

$$\dot{q} = \dot{q}_a + \int_{y_o}^{y_o - y} \tau \frac{\partial u_x}{\partial y} dy = \dot{q}_a + \dot{q}_j \tag{12}$$

In view of the approximations made Equation (12) can be applied with an uncertainty in the correction for viscous dissipation of not more than 5%, which is equivalent to an uncertainty in the final value of total conductivity of about 0.2%, which is much smaller than the experimental uncertainty of measurement of the total conductivity.

For steady, uniform flow the shear at any point may be evaluated from

$$\tau = \tau_o \left[\frac{y_o - 2y}{y_o} \right] \tag{13}$$

The shear at the wall may be evaluated from the pressure gradient and the distance between the plates as follows:

$$\tau_o = \frac{y_o}{2} \left(- \frac{dP}{dx} \right) \tag{14}$$

Equations (12), (13), and (14) may be combined to give

$$\dot{q}_j = \frac{1}{2} \left(- \frac{dP}{dx} \right) \int_{y_o}^{y_o - y} [y_o - 2y] \frac{du}{dy} dy \tag{15}$$

In the earlier evaluation of the experimental measurements (*13, 19*) values were reported in terms of eddy conductivity, which was calculated by determining the total conductivity and subtracting the molecular conductivity. Throughout this discussion the terms "corrected value" and "uncorrected value" are used to designate values of the total conductivity corrected and uncorrected for viscous dissipation.

The ratio of the total conductivity corrected for viscous dissipation to the uncorrected value is given by

$$\frac{[\epsilon_c + K]_{uncor}}{[\epsilon_c + K]_{cor}} = \frac{\dot{q}_a}{\dot{q}} = \frac{\dot{q}_a}{\dot{q}_a + \dot{q}_j} \tag{16}$$

For present purposes the published values (*13, 19*) were corrected for the effect of viscous dissipation as follows:

$$[\epsilon_c + K]_{cor} = \frac{\dot{q}}{\dot{q}_a} [\epsilon_c + K]_{uncor} = \left[1 + \frac{\dot{q}_j}{\dot{q}_a} \right] [\epsilon_c]_{uncor} \tag{17}$$

In carrying out these calculations the original values of thermal flux as determined from calorimetric measurements (*5, 18, 23*) were employed for evaluation of the term $\dot{q}_a$. The quantity $\dot{q}_j$ was established by application of Equation (15) with measured values of pressure difference (*23*) and velocity distribution (*18*). The average temperature gradients and Reynolds numbers were evaluated by the following equations:

$$\left(\frac{dt}{dy} \right)^* = \frac{t_a - t_i}{y_o} = \frac{\Delta t}{y_o} \tag{18}$$

$$N_{Re} = \frac{2}{\nu^*} \int_y^{y_o} u dy = \frac{2\, y_o\, U}{\nu^*} \tag{19}$$

The experimental conditions associated with the measurements under consideration have been reported (*13, 19*). An illustration of the uncorrected values of total conductivity (*13, 19*) is presented in the upper part of Figure 2. The data depicted are for a

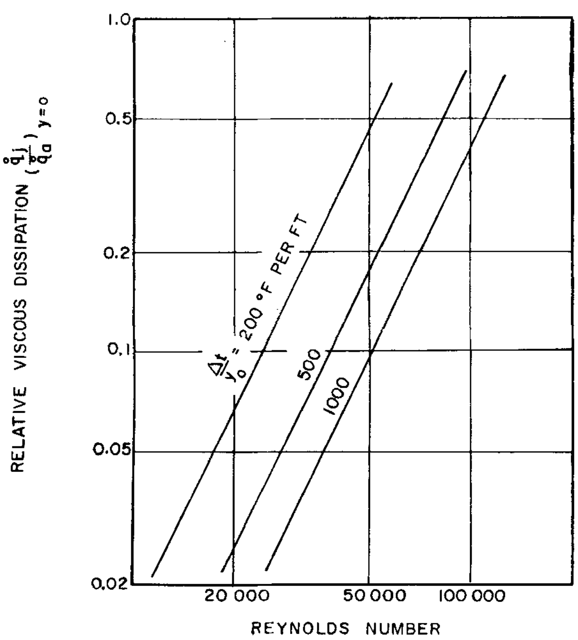

Fig. 4. Effect of conditions upon viscous dissipation at lower wall.

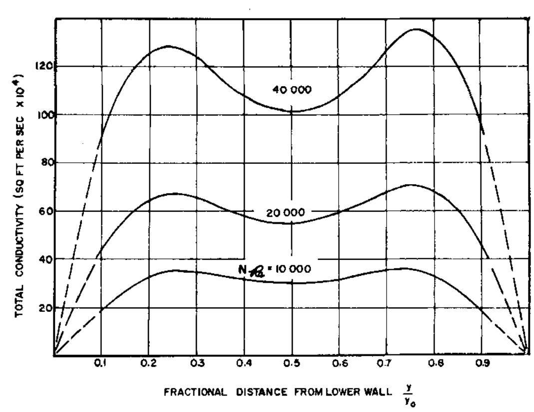

Fig. 5. Total conductivity as a function of position in channel.

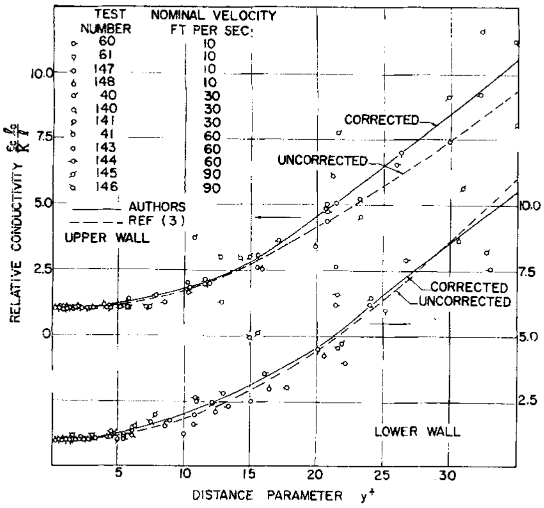

Fig. 6. Comparison of relative conductivity near upper and lower walls, corrected and uncorrected for viscous dissipation.

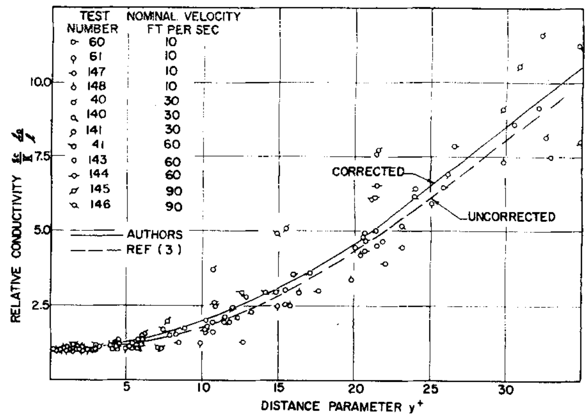

Fig. 7. Comparison of mean relative conductivity corrected and uncorrected for viscous dissipation.

Reynolds number, as defined in Equation (19), of approximately 54,000. It is apparent that the data are not symmetrical about the center line of the channel. A larger lack of symmetry occurs at the small average temperature gradients than at the large ones. At the time of the evaluation of these measurements it was thought that the lack of symmetry might be the result of reversal of the sign of the derivative of shear with respect to temperature on the two sides of the channel. However uncertainty concerning the lack of symmetry remained. The preponderance of experimental data was obtained in the upper half of the channel, and it was these data for the upper half of the channel which were used in the earlier evaluation of the effect of position and conditions of flow upon the total conductivity.

The data shown in the upper part of Figure 2 also indicate a rather large difference in the values of total conductivity obtained for different temperature gradients. The smallest values of total conductivity were obtained with the smallest temperature gradient. This effect was greater than the estimated experimental uncertainty. However for the values of average temperature gradient greater than about 500°F./ft., the effect of the gradient was small.

In the lower part of Figure 2 the corrected values of total conductivity are shown. It is evident that the symmetry of the data is improved and the effect of temperature gradient on the value of total conductivity is materially decreased. The amount of the correction for viscous dissipation is illustrated in Figure 3 as a function of position in the channel. The data are for a Reynolds number of 54,000. At the small temperature gradients the effect of viscous dissipation is greatest. It is apparent that at the top of the channel there is no effect and that the effect is at a maximum at the bottom of the channel. Figure 4 depicts the maximum correction as a function of average temperature gradient and Reynolds number.

The corrected values of total conductivity for air outside the boundary flows are presented in Table 1 as a function of position in the channel and Reynolds number. The values given represent the average of the results obtained for different values of the temperature gradient. There appeared to be no significant systematic variation in the total conductivity with temperature gradient after appropriate corrections for viscous dissipation were applied.

TABLE 3. RELATIVE CONDUCTIVITY CORRECTED FOR VISCOUS DISSIPATION

Distance parameter, y^+	Relative conductivity* Upper wall	Lower wall	Mean
1.0	1.010	1.015	1.015
2.0	1.035	1.050	1.050
3.0	1.080	1.100	1.100
4.0	1.135	1.165	1.165
5.0	1.205	1.250	1.250
6.0	1.290	1.365	1.360
7.0	1.385	1.500	1.495
8.0	1.500	1.660	1.645
9.0	1.630	1.850	1.810
10.0	1.780	2.050	1.995
11.0	1.945	2.255	2.185
12.0	2.140	2.465	2.390
13.0	2.340	2.670	2.595
14.0	2.560	2.885	2.820
15.0	2.795	3.120	3.060
16.0	3.065	3.370	3.320
17.0	3.375	3.640	3.595
18.0	3.745	3.925	3.890
19.0	4.140	4.235	4.215
20.0	4.540	4.560	4.560
21.0	4.945	4.945	4.945
σ†	0.5581	0.6021	0.5854

* Relative conductivity defined by $\epsilon_c l_o/Kl$.
† Standard error of estimate defined by $\sigma = \{[(\epsilon_c l_o/Kl)_s - (\epsilon_c l_o/Kl)_e]^2/(N-1)\}^{1/2}$.

Figure 5 depicts corrected values of total conductivity presented in Table 1. These values are consistent but somewhat higher than those reported earlier (*13, 19*). Near the center of the channel the effect of viscous dissipation is significant at the high Reynolds number. Furthermore there still exists some lack of symmetry. A comparison of the total conductivity corrected and uncorrected for viscous dissipation is presented in Table 2.

Cavers (*4*) reported values of relative conductivity as a function of the distance parameter y^+, which is defined as

$$y^+ = \frac{y_d}{\nu}\sqrt{\frac{\tau_{w}g}{\sigma}} \qquad (20)$$

By applying the corrections which have been discussed, the effect of viscous dissipation upon the values of relative conductivity in the boundary flow were established. Figure 6 shows the effect of the distance parameter y^+ upon experimental values of the relative conductivity corrected for viscous dissipation. The data illustrate the behavior near the upper and lower wall. The full curve shows the line of best fit to the data corrected for viscous dissipation, and the dashed curve represents the line of best fit to the uncorrected data (*4*). It is apparent that near the upper wall the effect of viscous dissipation is not large. In Figure 7 the mean values of relative conductivity are shown as a function of the distance parameter y^+. The corrected and uncorrected data again are represented by the full and dashed curves, respectively.

Table 3 records smooth values of the relative conductivity corrected for viscous dissipation as a function of the distance parameter y^+. Values are given for the lower wall, upper wall, and the mean. The same values of y^+ were used here as were originally chosen by Cavers (*4*). It has been established that the values of relative conductivity shown in Table 3 are consistent with the values of total conductivity shown in Table 2. The standard error of estimate of the experimental data from the curves shown in Figures 6 and 7 is given for the corrected relative conductivity. When the data are considered as a whole, there is little improvement in the agreement of the experimental points with the smooth curves as a result of the correction for viscous dissipation. The corrected data near the upper plate, where the correction is small, give a higher standard error of estimate than the uncorrected data. Near the lower plate however there is a marked improvement in the agreement of the corrected data with the smooth curve. Such behavior is to be expected, since the correction is most important near the lower wall.

DISCUSSION

In evaluating the utility of total conductivities it is well to remember that insofar as the experimental situation reflects the conditions under which the total conductivities are to be employed, the uncorrected values of total conductivity are useful. However in many situations the direction of the thermal flux may differ significantly from that in the experimental situation. Therefore the contribution of the viscous dissipation to the thermal flux, as evaluated from Equation (15) or other applicable relationships, would be interrelated with the total conductivity in a somewhat different fashion than that involved in the experimental work. Under such circumstances the use of total conductivity with appropriate correction for viscous dissipation is a proper approach to the problems associated with the temperature distribution and thermal fluxes in turbulent flow. It should be remembered that, as shown in Figures 3 and 4, the importance of the correction depends in large measure upon the Reynolds number of the flow, the local temperature gradient, and the position in the channel. The values recorded in Table 1 are not entirely symmetrical. The magnitude of this lack of symmetry is sufficient, together with the regularity of this behavior, to lend credence to the earlier supposition (13) that the lack of symmetry may be associated with the reversal of the sign of the derivative of shear with respect to temperature on the two sides of the channel.

ACKNOWLEDGMENT

The assistance of Virginia Berry in connection with some of the calculations associated with these corrections and of Ann Taylor in preparing the manuscript in a form suitable for publication is acknowledged. Emilio Venezian was the recipient of the Peter E. Fluor Memorial Fellowship. Grateful acknowledgment is made to the Fluor Foundation for the financial assistance which made this work possible.

NOTATION

C_P = isobaric heat capacity, B.t.u./(lb.)(°F.)
d = differential operator
E = energy of the turbulent motion, lb./sq.ft. $= \frac{P}{2} \sum_{i=1}^{3} (u'_i)^2$
g = acceleration of gravity, ft./sec.^{2}
k = thermal conductivity, B.t.u./(sq.ft.)(sec.)(°F./ft.)
l = distance from center line, ft.
l_o = distance from center line to wall, ft.
N = number of experimental points
N_{Re} = Reynolds number
$\dot{q}$ = thermal flux, B.t.u./(sq.ft.)(sec.)
$\dot{q}_j$ = local thermal flux due to viscous dissipation, B.t.u./(sq.ft.)(sec.)
P = pressure, lb./sq.ft.
P' = fluctuation in pressure, lb./sq.ft.
t = temperature, °F.
U = gross velocity, ft./sec. $= \frac{1}{y_o}\int_0^{y_o} u dy$
u = time-average local velocity, ft./sec.
u' = fluctuating component of local velocity, ft./sec.
x = distance along the axis of stream, ft.
y = distance normal to axis of stream measured from lower plate, ft.
y^+ = distance parameter
y_d = distance from nearer wall, ft.
y_o = separation between plates, ft.

Greek Letters

ϵ_c = eddy conductivity, sq.ft./sec.
$\underline{\epsilon}_c$ = total conductivity, sq.ft./sec.
η = absolute viscosity, lb.sec./sq.ft.
K = thermometric conductivity, sq.ft./sec. = $k/\sigma C_P$
ν = kinematic viscosity, sq.ft./sec.
ρ = density, lb.sec.^{2}/ft.^{4}
σ = specific weight, lb./cu.ft.
σ = standard error of estimate
τ = shear, lb./sq.ft.
τ_o = shear at wall, lb./sq.ft.
Φ = dissipation function, sec.$^{-2}$
Δ = change in

Subscripts

a = at upper plate
b = at lower plate
e = experimental value
i, j = dummy variables
m = molecular
s = smoothed value
t = turbulent
x = in the x direction
y = in the y direction

Superscript

$*$ = average value of

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Manuscript received March 6, 1961; revision received June 19, 1961; paper accepted June 21, 1961.